

Interior nodal sets of Steklov eigenfunctions on surfaces

Jiuyi Zhu

ABSTRACT. We investigate the interior nodal sets \mathcal{N}_λ of Steklov eigenfunctions on connected and compact surfaces with boundary. The optimal vanishing order of Steklov eigenfunctions is shown be $C\lambda$. The singular sets \mathcal{S}_λ are finite points on the nodal sets. We are able to prove that the Hausdorff measure $H^0(\mathcal{S}_\lambda) \leq C\lambda^2$. Furthermore, we obtain an upper bound for the measure of interior nodal sets $H^1(\mathcal{N}_\lambda) \leq C\lambda^{\frac{3}{2}}$. Here those positive constants C depend only on the surfaces.

1. Introduction

Let (\mathcal{M}, g) be a smooth, connected and compact surface with smooth boundary $\partial\mathcal{M}$. The main goal of this paper is to obtain an upper bound of interior nodal sets

$$\mathcal{N}_\lambda = \{z \in \mathcal{M} | e_\lambda = 0\}$$

for Steklov eigenfunctions

$$(1.1) \quad \begin{cases} \Delta_g e_\lambda = 0, & z \in \mathcal{M}, \\ \frac{\partial e_\lambda}{\partial \nu}(z) = \lambda e_\lambda(z), & z \in \partial\mathcal{M}, \end{cases}$$

where ν is a unit outward normal on $\partial\mathcal{M}$. The Steklov eigenfunctions were introduced by Steklov in 1902 for bounded domains in the plane. It interprets the steady state temperature distribution in the domain such that the heat flux on the boundary is proportional to the temperature. It is also found applications in a quite few physical fields, such as fluid mechanics, electromagnetism, elasticity, etc. Especially, the model (1.1) was studied by Calderón [C] as it can be regarded as eigenfunctions of the Dirichlet-to-Neumann map. The interior nodal sets of Steklov eigenfunctions represent the stationary points in \mathcal{M} . In the context of quantum mechanics, nodal sets are the sets where a free particle is least likely to be found.

It is well-known that the spectrum λ_j of Steklov eigenvalue problem is discrete with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3, \dots, \quad \text{and} \quad \lim_{j \rightarrow \infty} \lambda_j = \infty.$$

There exists an orthonormal basis $\{e_{\lambda_j}\}$ of eigenfunctions such that

$$e_{\lambda_j} \in C^\infty(\mathcal{M}), \quad \int_{\partial\mathcal{M}} e_{\lambda_j} e_{\lambda_k} dV_g = \delta_j^k.$$

1991 *Mathematics Subject Classification.* 35P20, 35P15, 58C40, 28A78.

Key words and phrases. Nodal sets, Upper bound, Steklov eigenfunctions.

Research is partially supported by the NSF grant DMS 1500468.

Estimating the Hausdorff measure of nodal sets has always been an important subject concerning the study of eigenfunctions. This subject centers around the famous Yau's conjecture. Recently, much work has been devoted to the bounds of nodal sets

$$Z_\lambda = \{z \in \partial\mathcal{M} | e_\lambda(z) = 0\}$$

of Steklov eigenfunctions on the boundary. Bellova and Lin [BL] proved the $H^{m-1}(Z_\lambda) \leq C\lambda^6$ with C depending only on \mathcal{M} , if \mathcal{M} is a $m+1$ dimensional analytic manifold. Zelditch [Z1] improved their results and gave the optimal upper bound $H^{m-1}(Z_\lambda) \leq C\lambda$ for analytic manifolds using microlocal analysis. For the smooth manifold \mathcal{M} , by assuming that 0 is a regular value, Wang and the author in [WZ] recently established a lower bound

$$H^{m-1}(Z_\lambda) \geq C\lambda^{\frac{3-m}{2}}.$$

Before presenting our results for interior nodal sets, let's briefly review the literature about the nodal sets of classical eigenfunctions. Interested reader may refer to the book [HL] and survey [Z] for detailed account about this subject. Let e_λ be L^2 normalized eigenfunctions of Laplacian-Beltrami operator on compact manifolds (\mathcal{M}, g) without boundary,

$$(1.2) \quad -\Delta_g e_\lambda = \lambda^2 e_\lambda.$$

Yau's conjecture states that for any smooth manifold, one should control the upper and lower bound of nodal sets of classical eigenfunctions as

$$(1.3) \quad c\lambda \leq H^{n-1}(\mathcal{N}_\lambda) \leq C\lambda$$

where C, c depends only on the manifold \mathcal{M} . The conjecture is only verified for real analytic manifolds by Donnelly-Fefferman in [DF]. Lin [Lin] also showed the upper bound for the analytic manifolds by a different approach. For the smooth manifolds, the conjecture is still not settled. For the lower bound of nodal sets with $n \geq 3$, Colding and Minicozzi [CM], Sogge and Zelditch [SZ], [SZ1] independently obtained that

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{\frac{3-n}{2}}$$

for smooth manifolds. See also [HSo] for deriving the same bound by adapting the idea in [SZ]. For the upper bound, Hardt and Simon [HS] gave an exponential upper bound

$$H^{n-1}(\mathcal{N}_\lambda) \leq Ce^{\lambda \ln \lambda}.$$

In surfaces, better results have been obtain. Brüning [Br] and Yau (unpublished) derived the same lower bound as (1.3). The best estimate to date for the upper bound is

$$H^1(\mathcal{N}_\lambda) \leq C\lambda^{\frac{3}{2}}$$

by Donnelly-Fefferman [DF2] and Dong [D] using different methods.

Let us return to Steklov eigenvalue problem (1.1). By the maximum principle, there exist nodal sets in the manifold \mathcal{M} and those sets must intersect the boundary $\partial\mathcal{M}$. Thus, it is natural to study the size of interior nodal sets in \mathcal{M} . We can also ask Yau's type questions about the Hausdorff measure of nodal sets. The natural and corresponding conjecture for Steklov eigenfunctions should states exactly the same as (1.3). See also the open questions in the survey by Girouard and Polterovich in [GP]. Recently, Sogge, Wang and the author [SWZ] obtained a lower bound for interior nodal sets

$$H^{n-1}(\mathcal{N}_\lambda) \geq C\lambda^{\frac{2-n}{2}}$$

for n -dimensional manifold \mathcal{M} . Very recently, Polterovich, Sher and Toth [PST] can verify Yau's type conjecture for (1.1) on real-analytic Riemannian surfaces.

An interesting topic related to the measure of nodal sets is about doubling inequality. Based on doubling inequalities, one can obtain the vanishing order of eigenfunctions, which characterizes how fast the eigenfunctions vanish. For the classical eigenfunctions of (1.2), Donnelly-Fefferman in [DF], [DF1] obtained that the maximal vanishing order of e_λ is at most order $C\lambda$ everywhere. To achieve it, a doubling inequality

$$(1.4) \quad \int_{\mathbb{B}(z_0, 2r)} e_\lambda^2 \leq C e^\lambda \int_{\mathbb{B}(z_0, r)} e_\lambda^2$$

is derived using Carleman estimates, where $\mathbb{B}(p, c)$ denotes as a ball centered at p with radius c . The doubling estimate (1.4) plays an important role in obtaining the bounds of nodal sets for analytic manifolds in [DF] and upper bound of nodal sets for smooth surfaces in [DF2]. For the Steklov eigenfunctions, we have obtained a doubling inequality on the boundary $\partial\mathcal{M}$ and derive that the sharp vanishing order is less than $C\lambda$ on the boundary $\partial\mathcal{M}$. For stecklov eigenfunction in \mathcal{M} , we are also able to get the doubling inequality, see proposition 1. With aid of doubling estimates and Carleman inequalities, The following optimal vanishing order for Steklov eigenfunctions can be obtained.

THEOREM 1. *The vanishing order of Steklov eigenfunction e_λ of (1.1) in \mathcal{M} is everywhere less than $C\lambda$.*

It's sharpness can be seen in the case that the manifold \mathcal{M} is a ball. Notice that the doubling estimates in proposition 1 and the vanishing order in Theorem 1 hold for any n -dimensional compact manifolds.

Singular sets

$$\mathcal{S}_\lambda = \{z \in \mathcal{M} | e_\lambda = 0, \nabla e_\lambda = 0\}$$

are contained in nodal sets. In Riemannian surfaces, those singular sets are finite points in the 1-dimensional nodal sets. It is interesting to count the number of those singular sets. Based on a Carleman inequality with singularities, we are able to show an upper bound of singular sets.

THEOREM 2. *Let (\mathcal{M}, g) be a smooth, compact surface with smooth boundary $\partial\mathcal{M}$. There holds*

$$(1.5) \quad H^0(\mathcal{S}_\lambda) \leq C\lambda^2$$

for Steklov eigenfunctions in (1.1).

For the nodal sets of Steklov eigenfunctions, we are able to build a similar type of Carleman inequality as [DF2] and show the following result.

THEOREM 3. *Let (\mathcal{M}, g) be a smooth, compact surface with smooth boundary $\partial\mathcal{M}$. Then*

$$(1.6) \quad H^1(\mathcal{N}_\lambda) \leq C\lambda^{\frac{3}{2}}$$

holds for Steklov eigenfunctions in (1.1).

The outline of the paper is as follows. Section 2 is devoted to reducing the Steklov eigenvalue problem into an equivalent elliptic equation without boundary. Then we obtain the optimal doubling inequality and show theorem 1. In section 3, we establish the Carleman inequality with singularities at finite points. Under additional assumptions of

those singular points, a stronger Carleman inequality is derived. We measure the singular sets in section 4. Section 5, 6 and 7 are devoted to obtaining the nodal length of Steklov eigenfunctions. Under the slow growth of L^2 norm condition, we find out the nodal length in section 6. Based on a similar type of Calderón and Zygmund decomposition procedure, we show the slow growth at almost every point. Then the measure of nodal sets is arrived by summing up the nodal length in each small square. The letter c, C, C_i, d_i denote generic positive constants and do not depend on λ . They may vary in different lines and sections.

Acknowledgement. It is my pleasure to thank Professor Christopher D. Sogge for helpful discussions about this topic and guidance into the area of eigenfunctions. I also would like to thank X. Wang for many fruitful conversations.

2. Vanishing Order of Steklov Eigenfunctions

In this section, we will reduce the Steklov eigenvalue problem to an equivalent model on a boundaryless manifold. The presence of eigenvalue on the boundary $\partial\mathcal{M}$ will be reflected on the coefficient functions of a second order elliptic equation. Let $d(z) = \text{dist}\{z, \partial\mathcal{M}\}$ denote the geodesic distance function from $x \in \mathcal{M}$ to the boundary $\partial\mathcal{M}$. Since \mathcal{M} is smooth, there exist a ρ -neighborhood of $\partial\mathcal{M}$ in \mathcal{M} such that $d(x)$ is smooth in the neighborhood. Let's denoted it as \mathcal{M}_ρ . We extend $d(z)$ smoothly in \mathcal{M} by

$$\delta(z) = \begin{cases} d(z) & z \in \mathcal{M}_\rho, \\ l(z) & z \in \mathcal{M} \setminus \mathcal{M}_\rho, \end{cases}$$

where $l(z)$ is a smooth function in $\mathcal{M} \setminus \mathcal{M}_\rho$. Note that the extended function $\delta(z)$ is a smooth function in \mathcal{M} . We first reduce Steklov eigenvalue problem into an elliptic equation with Neumann boundary condition. Let

$$v(z) = e_\lambda \exp\{\lambda\delta(z)\}.$$

It is known that $v(z) = e_\lambda(z)$ on $\partial\mathcal{M}$. For $z \in \partial\mathcal{M}$, $\nabla_g \delta(z) = -\nu(z)$. Recall that $\nu(z)$ is the unit outer normal on $z \in \partial\mathcal{M}$. We can check that the new function $v(z)$ satisfies

$$(2.1) \quad \begin{cases} \Delta_g v + b(z) \cdot \nabla_g v + q(z)v = 0 & \text{in } \mathcal{M}, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\mathcal{M}, \end{cases}$$

with

$$(2.2) \quad \begin{cases} b(z) = -2\lambda \nabla_g \delta(z), \\ q(z) = \lambda^2 |\nabla_g \delta(z)|^2 - \lambda \Delta_g \delta(z). \end{cases}$$

In order to get rid of boundary condition, we attach two copies of \mathcal{M} along the boundary and consider a double manifold $\overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{M}$. The metric g extends to $\overline{\mathcal{M}}$ with Lipschitz type singularity along $\partial\mathcal{M}$, since the lift metric g' of g on \mathcal{M} to the double manifold $\overline{\mathcal{M}}$ is Lipschitz. There also exists a canonical involutive isometry $\mathcal{F} : \overline{\mathcal{M}} \rightarrow \overline{\mathcal{M}}$ that interchanges the two copies of \mathcal{M} . Then the function $v(x)$ can be extended to $\overline{\mathcal{M}}$ by $v \circ \mathcal{F} = v$. Therefore, $v(z)$ satisfies

$$(2.3) \quad \Delta_{g'} v + \bar{b}(z) \cdot \nabla_{g'} v + \bar{q}(z)v = 0 \quad \text{in } \overline{\mathcal{M}}.$$

From (2.2), one can see that

$$(2.4) \quad \begin{cases} \|\bar{b}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda, \\ \|\bar{q}\|_{W^{1,\infty}(\overline{\mathcal{M}})} \leq C\lambda^2. \end{cases}$$

After this procedure, we can instead study the nodal sets for the second order elliptic equation (2.3) with assumption (2.4). Note that $\overline{\mathcal{M}}$ is a manifold without boundary.

We present a brief proof of Theorem 1. It is a small modification of the argument in [Zh], where the sharp vanishing order of Steklov eigenfunctions on boundary $\partial\mathcal{M}$ is shown to be less than $C\lambda$.

PROOF OF THEOREM 1. By a standard regularity argument, we can still consider polar coordinate for (2.3). We are able to establish a similar Carleman inequality as [Zh] for the general second order elliptic equation (2.3). See also e.g. [BC].

LEMMA 1. *Let $u \in C_0^\infty(\frac{1}{2}\epsilon_1 < r < \epsilon_0)$. If $\tau > C_1(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})$. Then*

$$(2.5) \quad \int r^4 e^{2\tau\phi(r)} |\Delta_{g'} u + \bar{b} \cdot \nabla_{g'} u + \bar{q} u|^2 dr d\omega \geq C_2 \tau^3 \int r^\epsilon e^{2\tau\phi(r)} u^2 dr d\omega,$$

where $\phi(r) = -\ln r + r^\epsilon$ and r is the geodesic distance. $0 < \epsilon_0, \epsilon_1, \epsilon < 1$ are some fixed constants. Moreover, (r, ω) are the standard polar coordinates.

Using this Carleman estimate and choosing suitable test functions, a Hadamard's three-ball result can be obtained in $\overline{\mathcal{M}}$. There exist constants r_0, C and $0 < \gamma < 1$ depending only on $\overline{\mathcal{M}}$ such that for any solutions of (2.3), $0 < r < r_0$, and $z_0 \in \overline{\mathcal{M}}$, one has

$$(2.6) \quad \int_{\mathbb{B}(z_0, r)} v^2 \leq e^{C(1 + \|\bar{b}\|_{W^{1,\infty}} + \|\bar{q}\|_{W^{1,\infty}}^{1/2})} \left(\int_{\mathbb{B}(z_0, 2r)} v^2 \right)^{1-\gamma} \left(\int_{\mathbb{B}(z_0, r/2)} v^2 \right)^\gamma.$$

Based on a propagation of smallness argument using the three-ball result and Carleman estimates (2.5), taking the assumptions (2.4) into account, we are able to obtain the doubling inequality in $\overline{\mathcal{M}}$.

PROPOSITION 1. *There exist constants r_0 and C depending only on $\overline{\mathcal{M}}$ such that for any $0 < r < r_0$ and $z_0 \in \overline{\mathcal{M}}$, there holds*

$$(2.7) \quad \|v\|_{L^2(\mathbb{B}(z_0, 2r))} \leq e^{C\lambda} \|v\|_{L^2(\mathbb{B}(z_0, r))}$$

for any solutions of (2.3).

One can see that the doubling estimate holds in \mathcal{M} if $\mathbb{B}(z_0, 2r) \subset \mathcal{M}$. By standard elliptic estimates, one can have L^∞ norm of doubling inequality

$$\|v\|_{L^\infty(\mathbb{B}(z_0, 2r))} \leq e^{C\lambda} \|v\|_{L^\infty(\mathbb{B}(z_0, r))}.$$

Since $\overline{\mathcal{M}}$ is compact, we can derive that

$$\|v\|_{L^\infty(\mathbb{B}(z_0, r))} \geq r^{C\lambda}$$

for any $z_0 \in \overline{\mathcal{M}}$, which implies the vanishing order for v is less than $C\lambda$. So is the vanishing order of u . This completes Theorem 1. \square

3. Carleman estimates

This section is devoted to establishing Carleman inequalities involving weighted functions at finite points. We will consider the behavior of v in a conformal coordinate patch. Since $\overline{\mathcal{M}}$ is a compact Riemannian surface. There exists a finite number N of conformal charts (\mathcal{U}_i, ϕ_i) with $\phi_i : \mathcal{U}_i \subset \overline{\mathcal{M}} \rightarrow \mathcal{V}_i \subset \mathbb{R}^2$ and $i \in \{1, 2, \dots, N\}$. On each of these charts, the metric is conformally flat and there exists positive function g_i such that $g' = g_i(x, y)(dx^2 + dy^2)$. By the compactness of the surface, there is positive constants c and C such that $0 < c < g_i < C$ for each i . Under this equivalent metric, the equation (2.3) can be written as

$$(3.1) \quad \Delta v + \bar{b}(z) \cdot \nabla v + \bar{q}(z)v = 0 \quad \text{in } \mathcal{V}_i,$$

where Δ is Euclidian Laplacian, ∇ is Euclidian gradient and $z = (x, y)$. We use the same notations $\bar{b}(z)$ and $\bar{q}(z)$ as that in (3.1), since they satisfy the same conditions as (2.4). They only differ by some function about g_i .

By restricting into a small ball $\mathbb{B}(p, 3c)$ contained in the conformal chart, we consider v in the small ball. Let $\bar{v}(z) = v(cz)$. It follows from (3.1) that

$$(3.2) \quad \Delta \bar{v} + \tilde{b}(z) \cdot \nabla \bar{v} + \tilde{q}(z)\bar{v} = 0 \quad \text{in } \mathbb{B}_3,$$

with $\tilde{b} = c\bar{b}$ and $\tilde{q} = c^2\bar{q}$. If c is sufficiently small, \tilde{b} and \tilde{q} are arbitrary small.

The crucial tool in [DF2] is a Carleman inequality for classical eigenfunctions involving weighted functions with singularity at finite points. We will obtain the corresponding Carleman inequality for the second order elliptic equation (3.2). We adapt the approach in [DF2] to obtain the desirable Carleman estimate for (3.2).

Let $\mathcal{D} \subset \mathcal{C}$ be an open set and $\psi \in C_0^\infty(\mathcal{D})$ be a real valued uncton. We introduce the following differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Direct computation shows that $\bar{\partial}\partial\psi = \frac{1}{4}\Delta\psi$. By the Cauchy-Riemann equation, u is holomorphic if and only if $\bar{\partial}u = 0$. For the completeness, we present the elementary inequality in [DF2].

LEMMA 2. *let Φ be a smooth positive function in \mathcal{D} . Then*

$$(3.3) \quad \int_{\mathcal{D}} |\bar{\partial}u|^2 \Phi \geq \frac{1}{4} \int_{\mathcal{D}} (\Delta \ln \Phi) |u|^2 \Phi.$$

Here the integral is taken with respect to the lebesgue measure.

We want the weight function to involve those singular points. To specialize the choice of Φ , we construct the following function ψ_0 .

LEMMA 3. *There exists a smooth function ψ_0 defined for $|z| > 1 - 2a$ satisfying the following properties:*

- (i) $a_1 \leq \psi_0(z) \leq a_2$ with constant $a_1, a_2 > 0$.
- (ii) $\psi_0 = 1$ on $\{|z| > 1\}$.
- (iii) $\Delta \ln \psi_0 \geq 0$ on $\{|z| > (1 - 2a)\}$.
- (iv) If $1 - 2a < |z| < 1 - a$, then $\Delta \ln \psi_0 \geq a_3 > 0$.

The existence of such ψ_0 follows from existence and unique theory of ordinary differential equations.

We assume that

$$D_l = \{z \mid |z - z_l| \leq \delta\}.$$

Let D_l be a finite collection of pairwise disjoint disks, which are contained in a unit disk centered at origin. Let

$$D_l(a) = \{z \mid |z - z_l| \leq (1 - 2a)\delta\}$$

be the smaller concentric disk. We define a smooth weight function $\Psi_0(z)$ as

$$\Psi_0(z) = \begin{cases} 1 & \text{if } z \notin \cup_l D_l, \\ \psi_0(\frac{z-z_l}{\delta}) & \text{if } z \in D_l. \end{cases}$$

We also introduce the following domain

$$A_l = \{(1 - 2a)\delta \leq |z - z_l| \leq (1 - a)\delta\}.$$

From the last lemma, $\Psi_0(z)$ satisfies those properties:

- (i) $a_1 \leq \Psi_0(z) \leq a_2$.
- (ii) $\Delta \ln \Psi_0 \geq 0$ for $z \in \mathbb{R}^2 \setminus \cup_l D_l(a)$.
- (iii) $\Delta \ln \Psi_0 \geq a_3 \delta^{-2}$ for $z \in A_l$.

Note that a_i in above are positive constants independent of λ . Denote

$$A = \cup_l A_l.$$

Suppose that τ is a nonnegative constant. We introduce $\Phi(z) = \Psi_0(z)e^{\tau|z|^2}$. For $u \in C_0^\infty(\mathbb{R}^2 \setminus \cup_l D_l(a))$, we assume that \mathcal{D} contains the support of u and $A \subset \mathcal{D} \subset \mathbb{R}^2 \setminus \cup_l D_l(a)$. Obviously,

$$\ln \Phi(z) = \ln \Psi_0(z) + \tau|z|^2.$$

Substituting Φ in lemma 2 gives that

$$(3.4) \quad \int_{\mathcal{D}} |\bar{\partial} u|^2 \Psi_0(z) e^{\tau|z|^2} \geq C_1 \tau \int_{\mathcal{D}} |u|^2 \Psi_0(z) e^{\tau|z|^2} + C_2 \delta^{-2} \int_A |u|^2 e^{\tau|z|^2},$$

where we have used the properties (ii) and (iii) for Ψ_0 . The boundedness of $\Psi_0(z)$ yields that

$$(3.5) \quad \int_{\mathcal{D}} |\bar{\partial} u|^2 e^{\tau|z|^2} \geq C_3 \tau \int_{\mathcal{D}} |u|^2 e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |u|^2 e^{\tau|z|^2}.$$

Define a holomorphic function

$$P(z) = \prod_l (z - z_l).$$

Then $\bar{\partial}(u/P) = \bar{\partial}u/P$. Replacing u by u/P in (3.5), it follows that

$$(3.6) \quad \int_{\mathcal{D}} |\bar{\partial} u|^2 |P|^{-2} e^{\tau|z|^2} \geq C_3 \tau \int_{\mathcal{D}} |u|^2 |P|^{-2} e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |u|^2 |P|^{-2} e^{\tau|z|^2}.$$

We will establish a Carleman inequality for second order elliptic equations as (3.2). Write $\tilde{b}(x) = (\tilde{b}_1(x), \tilde{b}_2(x))$. Let

$$u = \partial f + \frac{1}{2}(\tilde{b}_1 - i\tilde{b}_2)f,$$

where $f \in C_0^\infty(\mathbb{R}^2 \setminus \cup_l D_l(a))$ is a real valued function. Then

$$\bar{\partial}u = \frac{1}{4}[\Delta f + \operatorname{div} \tilde{b}f + \tilde{b} \cdot \nabla f + i(\frac{\partial(\tilde{b}_1 f)}{\partial y} - \frac{\partial(\tilde{b}_2 f)}{\partial x})].$$

Plugging above u into (3.6), we obtain

$$\begin{aligned} \int_{\mathcal{D}} [|\Delta f + \tilde{b} \cdot \nabla f|^2 + |\operatorname{div} \tilde{b}f|^2 + |\frac{\partial(\tilde{b}_1 f)}{\partial y} - \frac{\partial(\tilde{b}_2 f)}{\partial x}|^2] |P|^{-2} e^{\tau|z|^2} \\ \geq C_3 \tau \int_{\mathcal{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} - C_3 \tau \int_{\mathcal{D}} |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2} \\ + C_4 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} - C_4 \delta^{-2} \int_A |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned} \quad (3.7)$$

If we choose $u = f$ in (3.6), we get

$$\int_{\mathcal{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_3 \tau \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{\tau|z|^2}. \quad (3.8)$$

Since the norm of \tilde{b} is chosen small enough, it is smaller than τ which will be chosen large enough. With aid of (3.8), we can incorporate the terms about \tilde{b} in the left hand side of (3.7) into the first term in the right hand side of (3.7),

$$\begin{aligned} \int_{\mathcal{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_5 \tau \int_{\mathcal{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} + C_4 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} \\ - C_4 \delta^{-2} \int_A |\tilde{b}|^2 |f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned} \quad (3.9)$$

Furthermore, if $u = f$, the inequality (3.6) implies that

$$\int_{\mathcal{D}} |\nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_4 \delta^{-2} \int_A |f|^2 |P|^{-2} e^{\tau|z|^2}. \quad (3.10)$$

Applying (3.10) to the last term in the right hand side of (3.9) gives that

$$\begin{aligned} \int_{\mathcal{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C_6 \tau^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{\tau|z|^2} \\ + C_7 \delta^{-2} \int_A |\nabla f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned} \quad (3.11)$$

We continue to get a refined estimate for the last term of (3.11). In order to achieve this goal, we need the following hypotheses for the geometry of the disk D_l and the parameter $\tau > 1$:

(R_1) The radius δ for each disk D_l is less than $a_4 \tau^{-1}$.

(R_2) The distance between any two distinct z_l is at least $2a_5 \tau^{\frac{1}{2}} \delta$.

(R_3) The total number of disk D_l is at most $a_6 \tau$.

Under the those assumptions, we have those comparison estimates from [DF2].

LEMMA 4. *If \bar{z}_1 and \bar{z}_2 are any points in the same component A_l of A , then*

(i) $a_7 < e^{\tau|\bar{z}_1|^2} / e^{\tau|\bar{z}_2|^2} < a_8$.

(ii) $a_9 < |P(\bar{z}_1)| / |P(\bar{z}_2)| < a_{10}$.

We also need the following Poincaré type inequality on each annulus. If $f \in C^\infty(A_l)$ and f vanishes on the inner boundary of A_l , then

$$(3.12) \quad \int_{A_l} |\nabla f|^2 \geq a_{11} \delta^{-2} \int_{A_l} |f|^2.$$

The proof of (3.12) can be seen in [DF2]. Let $z_l \in A_l$ be chosen arbitrarily. By lemma 4, it follows that

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_8 \sum_l e^{\tau|z_l|^2} |P(z_l)|^{-2} \int_{A_l} |\nabla f|^2.$$

Since $f \in C_0^\infty(\mathbb{R}^2 \setminus \cup_l D_l(a))$, the inequality (3.12) yields that

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_9 \sum_l e^{\tau|z_l|^2} |P(z_l)|^{-2} \delta^{-2} \int_{A_l} |f|^2.$$

Using lemma 4 again, we obtain

$$\int_{A_l} |\nabla f|^2 |P(z)|^{-2} e^{\tau|z|^2} \geq C_{10} \delta^{-2} \int_A |f|^2 |P(z)|^{-2} e^{\tau|z|^2}.$$

Substituting the last inequality into the last term in (3.11) leads to

$$(3.13) \quad \begin{aligned} \int_{\mathcal{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} &\geq C_6 \tau^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{\tau|z|^2} \\ &+ C_{11} \delta^{-4} \int_A |f|^2 |P|^{-2} e^{\tau|z|^2}. \end{aligned}$$

We summarize the above arguments in the following proposition.

PROPOSITION 2. *Assuming $f \in C_0^\infty(\mathbb{R}^2 \setminus \cup_l D_l(a))$. Then*

(i) *it holds that*

$$(3.14) \quad \int_{\mathcal{D}} |\Delta f + \tilde{b} \cdot \nabla f|^2 |P|^{-2} e^{\tau|z|^2} \geq C \tau^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{\tau|z|^2}.$$

(ii) *If the additional assumptions (R_1) – (R_3) for D_l hold, the stronger inequality (3.13) is satisfied.*

4. Measure of Singular Sets

Let \mathcal{M} be a compact smooth surface. In section 2, we have shown that the Steklov eigenfunction e_λ vanishes at any points at most $C\lambda$. By the implicit function theorem, outside the singular sets, the nodal set is locally a 1-dimensional C^1 manifold. Adapting the arguments in [DF2] for (3.2), we can estimate those singular points in a quantitative way. We are able to obtain an upper bound for the singular points in term of eigenvalue λ .

LEMMA 5. *Singular sets consist of at most finite many points.*

PROOF. Without loss of generality, we assume that $0 \in \mathcal{S}_\lambda$ and choose normal coordinate (x, y) at the origin. Next we prove there are finite singular points in $\overline{\mathcal{M}}$. Using Taylor expansion, we expand v locally at origin. Then $v(x, y) = F_j(x, y) + W_{j+1}(x, y)$, where $F_j(x, y)$ consists of the leading nonvanishing term with homogenous order $j \geq 2$. $W_{j+1}(x, y)$ is a higher order reminder term. Since $\Delta v + \tilde{b}(z) \cdot \nabla v + \bar{q}(z)v = 0$ and the coordinate is normal, we obtain that $\Delta F_j = 0$. Under polar coordinates, we find that

$F_j = r^j (a_1 \cos(j\theta) + a_2 \sin(j\theta))$. Obviously, $r^{-1} \frac{\partial F_j}{\partial \theta}$ and $\frac{\partial F_j}{\partial r}$ have no common zero if $r \neq 0$. Since

$$|\nabla F_j|^2 = \left| \frac{\partial F_j}{\partial r} \right|^2 + \frac{1}{r^2} \left| \frac{\partial F_j}{\partial \theta} \right|^2,$$

there exists a small neighborhood of \mathcal{U} of origin such that $\mathcal{U} \cap \mathcal{S}_\lambda = \emptyset$. Since $\overline{\mathcal{M}}$ is compact, then the lemma follows. \square

We plan to count the number of singular points in a sufficiently small ball. Let $p \in \overline{\mathcal{M}}$. Consider a geodesic ball $\mathbb{B}(p, c\lambda^{-\frac{1}{2}})$. If c is small enough, then this geodesic ball is contained in a conformal chart. If we choose

$$w(z) = v(c\lambda^{-\frac{1}{2}}z)$$

with c sufficiently small. From equation (3.1), w satisfies

$$(4.1) \quad \Delta w + \hat{b}(x) \cdot \nabla w + \hat{q}(x)w = 0 \quad \text{in } \mathbb{B}(0, 4),$$

with $\hat{b}(x) = c\lambda^{-\frac{1}{2}}\bar{b}(x)$ and $\hat{q}(x) = c^2\lambda^{-1}\bar{q}(x)$. From (2.4), we obtain

$$(4.2) \quad \begin{cases} \|\hat{b}\|_{W^{1,\infty}(\mathbb{B}(0,4))} \leq c\lambda^{\frac{1}{2}}, \\ \|\hat{q}\|_{W^{1,\infty}(\mathbb{B}(0,4))} \leq c^2\lambda \end{cases}$$

with c sufficiently small.

Next we will count the total order of vanishing of singular points for w in the sufficiently small ball. We study w in equation (4.1),

PROPOSITION 3. *Suppose $z_l \in \mathcal{S}_\lambda \cap \mathbb{B}(p, c\lambda^{-\frac{1}{2}})$ where v vanishes to order $n_l + 1$. Then $\sum_l n_l \leq C\lambda$.*

PROOF. It suffices to count the number of singular point of w in a small Euclidean ball with radius $\frac{1}{10}$ centered at origin. Suppose that w vanishes to order $n_l + 1$. Let $n_l = m_l + 1$. We first consider the case $n_l \geq 2$. Then $m_l \geq 1$. Define the polynomial

$$P(z) = \prod (z - z_l)^{m_l}$$

with $|z_l| < \frac{1}{10}$. Let $\mathcal{D} = \mathbb{B}(0, 2)$ and D_l be small disjoint disks of radius δ centered at z_l . If $f \in C_0^\infty(\mathbb{R}^2 \setminus \cup_l D_l)$, the inequality (3.14) in proposition 2 implies that

$$(4.3) \quad \int_{\mathcal{D}} (|\Delta f|^2 + |\hat{b} \cdot \nabla f|^2) |P|^{-2} e^{d_1 \lambda |z|^2} \geq C_2 \lambda^2 \int_{\mathcal{D}} |f|^2 |P|^{-2} e^{d_1 \lambda |z|^2},$$

where $\tau = d_1 \lambda$. We choose a cut-off function $\theta(z)$ such that θw is compact support in \mathcal{D} . We select the cut-off function $\theta \in C_0^\infty(\mathcal{D} \setminus \cup_l D_l)$ with following properties:

- (i) $\theta(z) = 1$, if $|z| < \frac{3}{2}$ and $|z - z_l| > 2\delta$.
- (ii) $|\nabla \theta| < C_3$, $|\Delta \theta| < C_4$ if $|z| > \frac{3}{2}$.
- (iii) $|\nabla \theta| < C_5 \delta^{-1}$, $|\Delta \theta| < C_6 \delta^{-2}$ if $|z - z_l| < 2\delta$.

Substituting $f = \theta w$ into (4.3) yields that

$$\int_{(|z| < \frac{3}{2}) \cup (\frac{3}{2} \leq |z| \leq 2)} |\Delta(\theta w) + \tilde{b} \cdot \nabla(\theta w)|^2 |P|^{-2} e^{d_1 \lambda |z|^2} \geq C_2 \lambda^2 \int_{|z| < \frac{3}{2}} |w|^2 |P|^{-2} e^{d_1 \lambda |z|^2}.$$

From the equation (4.1),

$$\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w) = -\hat{q}\theta w + \Delta\theta w + 2\nabla\theta \cdot \nabla w + \hat{b} \cdot \nabla\theta w.$$

By the assumption of θ , we obtain

$$|\Delta\theta w| + |\nabla\theta \cdot \nabla w| + |\nabla\theta w| \leq C_7\delta^{m_l}, \text{ if } |z - z_l| \leq 2\delta.$$

Taking $\delta \rightarrow 0$, by dominated convergence theorem, we have

$$(4.4) \quad \begin{aligned} c\lambda^2 \int_{|z| < \frac{3}{2}} |w|^2 |P|^{-2} e^{d_1\lambda|z|^2} + C_9(1+\lambda)^2 \int_{\frac{3}{2} \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) |P|^{-2} e^{d_1\lambda|z|^2} \\ \geq C_{10}\lambda^2 \int_{|z| < \frac{3}{2}} |w|^2 |P|^{-2} e^{d_1\lambda|z|^2}. \end{aligned}$$

Since c is sufficiently small, we can absorb the first term in the left hand side of (4.4) into the right hand side. Then

$$(4.5) \quad \int_{\frac{3}{2} \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) |P|^{-2} e^{d_1\lambda|z|^2} \geq C_{11} \int_{|z| \leq \frac{1}{2}} |w|^2 |P|^{-2} e^{d_1\lambda|z|^2}.$$

Obviously, it follows that

$$(4.6) \quad \max_{|z| \geq \frac{3}{2}} |P|^{-2} \int_{\frac{3}{2} \leq |z| \leq 2} (|w|^2 + |\nabla w|^2) e^{d_1\lambda|z|^2} \geq C_{11} (\min_{|z| \leq \frac{1}{2}} |P|^{-2}) \int_{|z| \leq \frac{1}{2}} |w|^2.$$

By standard elliptic theory, the last inequality implies

$$(4.7) \quad \max_{|z| \geq \frac{3}{2}} |P|^{-2} e^{d_2\lambda} \int_{|z| \leq \frac{5}{2}} |w|^2 \geq C_{11} (\min_{|z| \leq \frac{1}{2}} |P|^{-2}) \int_{|z| \leq \frac{1}{2}} |w|^2.$$

We claim that

$$(4.8) \quad e^{d_3 \sum m_l} \leq \frac{\min_{|z| \leq \frac{1}{2}} |P|^{-2}}{\max_{|z| \geq \frac{3}{2}} |P|^{-2}}.$$

To prove (4.8), it reduces to verify

$$(4.9) \quad e^{-d_4 \sum m_l} \min_{|z| \geq \frac{3}{2}} |P| \geq \max_{|z| \leq \frac{1}{2}} |P|.$$

away from singular point z_l . Clearly,

$$\max_{|z| \leq \frac{1}{2}} |P| \leq \left(\frac{1}{2}\right)^{\sum m_l}.$$

Since $z_l \in \mathbb{B}(0, \frac{1}{10})$, we have

$$\left(\frac{3}{4}\right)^{\sum m_l} \leq \min_{|z| \geq \frac{3}{2}} |P|.$$

Combining the last two inequalities, we obtain (4.9). The claim is shown. Let's return to (4.7), we get

$$(4.10) \quad \begin{aligned} \frac{\min_{|z| \leq \frac{1}{2}} |P|^{-2}}{\max_{|z| \geq \frac{3}{2}} |P|^{-2}} &\leq \frac{e^{d_5\lambda} \int_{|z| \leq \frac{5}{2}} |w|^2}{C_{11} \int_{|z| \leq \frac{1}{2}} |w|^2} \\ &\leq e^{d_6\lambda}, \end{aligned}$$

where we applied doubling estimates in the last inequality. Thanks to (4.8), we obtain

$$\sum m_l \leq d_7\lambda.$$

Since $n_l = m_l + 1 \leq 2m_l$, we complete the lemma for $n_l \geq 2$.

If the vanishing order for the singular point is two, i.e. $n_l = 1$. We consider $Q(z) = \prod (z - z_l)^{\frac{n_l}{2}}$ instead of $P(z)$. In this case, $Q(z)$ may not be defined as a single valued holomorphic function on \mathcal{C} . We pass to a finite branched cover of the disk \mathcal{D} punctured at z_l . The Carleman estimates in previous sections still work. The same conclusion will follow. \square

Based on the vanishing order estimate in proposition 3, we are able to count the number of singular points.

PROOF OF THEOREM 2. We cover the double manifold $\overline{\mathcal{M}}$ by geodesic balls with radius $C\lambda^{-1/2}$. Since $\overline{\mathcal{M}}$ is compact, the order of those balls is $C\lambda$. From proposition 3, the conclusion in theorem 2 is arrived. \square

REMARK 1. *Thanks to proposition 3, we can actually show a stronger result. Let $z_l \in \mathcal{M}$ be singular point with vanishing order $n_l + 1$, Then $\sum_l n_l \leq C\lambda^2$.*

5. Growth of eigenfunctions

In this section, we will show that the eigenfunctions do not grow rapidly on too many small balls. We still restrict v into the small geodesic ball $\mathbb{B}(p, c\lambda^{-\frac{1}{2}})$ in the conformal chart. Let $w(z) = v(c\lambda^{-\frac{1}{2}}z)$. Then w satisfies the elliptic equation (4.1) with assumptions (4.2) in a Euclidean ball of radius four centered at origin. If we suppose that w grow rapidly, that is,

$$(5.1) \quad C_1 \int_{(1-3a)\delta \leq |z-z_l| \leq (1-\frac{3a}{4})\delta} w^2 \leq \int_{(1-\frac{3a}{2})\delta \leq |z-z_l| \leq (1-a)\delta} w^2$$

for all l and some large C_1 , then the following proposition is valid.

PROPOSITION 4. *Suppose D_l are disks contained in a Euclidean ball of radius $\frac{1}{30}$ centered at origin. Furthermore, assume that (R_1) : $\delta < d_1\lambda^{-1}$ and (R_2) : $|z_l - z_k| > d_2\lambda^{\frac{1}{2}}\delta$, when $l \neq k$. If (5.1) holds for all l , the number of disks D_l is less than $d_3\lambda$.*

PROOF. We will use the stronger Carleman estimates in (3.13) in proposition 2. We prove it by contradiction. Suppose that the collection $D_l = \{z \mid |z - z_l| \leq \delta\}$ are disjoint disks satisfying the hypotheses (R_1) – (R_3) in section 3. Without loss of generality, we require all the D_l are in a ball centered at origin with radius $\frac{1}{30}$. As before, $D_l(a) = \{z \mid |z - z_l| \leq (1 - 2a)\delta\}$, where a is a suitably small positive constant. Let \mathcal{D} be a ball centered at origin with radius 2. We choose a cut-off function $\theta \in C_0^\infty(\mathcal{D} \setminus \cup_l D_l)$ and assume $\theta(z)$ satisfies the following properties:

(i) $\theta(z) = 1$, if $|z| < 1$ and $|z - z_l| > (1 - \frac{3}{2}a)\delta$ for all l .

(ii) $|\nabla\theta| + |\Delta\theta| < C_2$ if $|z| > 1$.

(iii) $|\nabla\theta| < C_3\delta^{-1}$, $|\Delta\theta| < C_4\delta^{-2}$ if $|z - z_l| < (1 - \frac{3}{2}a)\delta$.

Substituting $f = \theta w$ into (3.13) gives that

$$(5.2) \quad \begin{aligned} \int_{\mathcal{D}} |\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w)|^2 |P|^{-2} e^{d_4\lambda|z|^2} &\geq C_5\lambda^2 \int_{\mathcal{D}} |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2} \\ &+ C_6\delta^{-4} \int_A |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned}$$

We also assume $\tau = d_4\lambda$. Recall that $A = \cup_l A_l$ and $A_l = \{z | (1-2a)\delta \leq |z-z_l| \leq (1-a)\delta\}$. We first consider the integral in the left hand side of the last inequality. Again, by (4.1),

$$\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w) = -\hat{q}\theta w + \Delta\theta w + 2\nabla\theta \cdot \nabla w + \hat{b} \cdot \nabla\theta w.$$

Thus,

$$|\Delta(\theta w) + \hat{b} \cdot \nabla(\theta w)|^2 \leq C(c\lambda^2\theta^2w^2 + |\Delta\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2 + c\lambda|\nabla\theta|^2w^2),$$

where c is sufficiently small. We will absorb the term involving θ^2w^2 into the right hand side of (5.2). Since c is small enough, we get

$$(5.3) \quad \begin{aligned} \int_{\mathcal{D}} (|\Delta\theta|^2w^2 + c|\nabla\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2} &\geq C_7\lambda^2 \int_{\mathcal{D}} |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2} \\ &+ C_8\delta^{-4} \int_A |\theta w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned}$$

Using the properties of $\theta(z)$ and taking into account that all D_l lies in the ball centered at origin with radius $\frac{1}{30}$, we obtain

$$(5.4) \quad \begin{aligned} \int_{\mathcal{D}} (|\Delta\theta|^2w^2 + c|\nabla\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2) |P|^{-2} e^{d_4\lambda|z|^2} &\geq C_7\lambda^2 \int_{\frac{1}{4} \leq |z| \leq \frac{1}{2}} |w|^2 |P|^{-2} e^{d_4\lambda|z|^2} \\ &+ C_9\delta^{-4} \sum_l \int_{(1-\frac{3a}{2})\delta \leq |z-z_l| \leq (1-a)\delta} |w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned}$$

Next we want to control the left hand side of last inequality. Write

$$(5.5) \quad \int_{\mathcal{D}} |\Delta\theta|^2w^2 + c|\nabla\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2 |P|^{-2} e^{d_4\lambda|z|^2} = I + \sum_l I_l,$$

where

$$\begin{aligned} I &= \int_{1 \leq |z| \leq 2} |\Delta\theta|^2w^2 + c|\nabla\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2 |P|^{-2} e^{d_4\lambda|z|^2}, \\ I_l &= \int_{(1-2a)\delta \leq |z-z_l| \leq (1-\frac{3a}{2})\delta} |\Delta\theta|^2w^2 + c|\nabla\theta|^2w^2 + |\nabla\theta|^2|\nabla w|^2 |P|^{-2} e^{d_4\lambda|z|^2}. \end{aligned}$$

By standard elliptic estimates,

$$(5.6) \quad I \leq e^{d_5\lambda} \max_{|z| \geq 1} |P|^{-2} \int_{\frac{3}{4} \leq |z| \leq \frac{5}{2}} w^2.$$

Similarly, via elliptic estimates,

$$(5.7) \quad I_l \leq C_{10}\delta^{-4} (\max_{A_l} |P|^{-2} e^{d_4\lambda|z|}) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-\frac{3a}{4})\delta} w^2.$$

Thanks to lemma 4,

$$(5.8) \quad I_l \leq C_{11}\delta^{-4} (\min_{A_l} |P|^{-2} e^{d_4\lambda|z|}) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-\frac{3a}{4})\delta} w^2.$$

Combining those inequalities together in (5.4) leads to

$$\begin{aligned}
(5.9) \quad e^{d_5\lambda} \max_{|z| \geq 1} |P|^{-2} \int_{\frac{3}{4} \leq |z| \leq \frac{5}{2}} w^2 &+ C_{11} \delta^{-4} \sum_l (\min_{A_l} |P|^{-2} e^{d_4\lambda|z|}) \int_{(1-3a)\delta \leq |z-z_l| \leq (1-\frac{3a}{4})\delta} w^2 \\
&\geq C_{12} \min_{|z| \leq \frac{1}{2}} |P|^{-2} \int_{\frac{1}{4} \leq |z| \leq \frac{1}{2}} |w|^2 \\
&+ C_{13} \delta^{-4} \sum_l \min_{A_l} (|P|^{-2} e^{d_4\lambda|z|^2}) \int_{(1-\frac{3a}{2})\delta \leq |z-z_l| \leq (1-a)\delta} w^2.
\end{aligned}$$

Performing the similar arguments as (4.8) shows that

$$\min_{|z| \leq \frac{1}{2}} |P|^{-2} > \max_{|z| \geq 1} |P|^{-2} e^{d_5 \sum_l m_l}.$$

If the number of D_l is $d_3\lambda$, then

$$(5.10) \quad \min_{|z| \leq \frac{1}{2}} |P|^{-2} > \max_{|z| \geq 1} |P|^{-2} e^{d_6\lambda}.$$

We claim that

$$(5.11) \quad e^{C_{14}\lambda} \int_{\frac{1}{4} \leq |z| \leq \frac{1}{2}} w^2 \geq \int_{\frac{3}{4} \leq |z| \leq \frac{5}{2}} w^2.$$

We prove the claim by doubling estimates shown in proposition 1. We choose a ball $\mathbb{B}(x_0, \frac{1}{8}) \subset \{z | \frac{1}{4} \leq |z| \leq \frac{1}{2}\}$. It is clear that

$$\int_{\frac{1}{4} \leq |z| \leq \frac{1}{2}} w^2 \geq \int_{\mathbb{B}(x_0, \frac{1}{8})} w^2.$$

Using doubling estimates, we have

$$e^{C_{15}\lambda} \int_{\mathbb{B}(x_0, \frac{1}{8})} w^2 \geq \int_{\mathbb{B}(x_0, \frac{2}{8})} w^2.$$

By finite iterations, we can find a large ball $\mathbb{B}(x_0, 3)$ that contains $\{z | \frac{3}{4} \leq |z| \leq \frac{5}{2}\}$. It yields that

$$\int_{\mathbb{B}(x_0, 3)} w^2 \geq \int_{\frac{3}{4} \leq |z| \leq \frac{5}{2}} w^2.$$

Then the combination of those inequalities verify the claim.

If we choose d_3 is suitably large, since the number disk D_l is $d_3\lambda$, then d_6 is suitably large. From the inequalities (5.10) and (5.11), it follows that

$$(5.12) \quad e^{d_5\lambda} \max_{|z| \geq 1} |P|^{-2} \int_{\frac{3}{4} \leq |z| \leq \frac{5}{2}} w^2 < C_{12} \min_{|z| \leq \frac{1}{2}} |P|^{-2} \int_{\frac{1}{4} \leq |z| \leq \frac{1}{2}} w^2.$$

It contradicts the estimates (5.1) and (5.9). The proposition is arrived. \square

6. Growth Estimates and Nodal Length

This section is to find the connection between growth of eigenfunctions and nodal length. A suitable small growth in L^2 norm implies an upper bound of nodal length. We consider the second order elliptic equations

$$(6.1) \quad \Delta \bar{w} + b^* \cdot \nabla \bar{w} + q^* \bar{w} = 0 \quad \text{in } \mathbb{B}(0, 4).$$

Assume that there exist a positive constant C such that $\|b^*\|_{W^{1,\infty}} \leq C$ and $\|q^*\|_{W^{1,\infty}} \leq C$. The following lemma relies on the Carleman estimates in lemma 1. Suppose ϵ_1 is a sufficiently small positive constant.

LEMMA 6. *Suppose that w satisfies the growth estimate*

$$(6.2) \quad \int_{(1-\frac{3a}{2})\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 \leq C_3 \int_{(1-3a)\epsilon_0 < r < (1-\frac{4a}{3})\epsilon_0} \bar{w}^2,$$

where a and ϵ_0 are fixed small constants. Then for $0 < \epsilon_1 < \frac{\epsilon_0}{100}$, we have

$$(6.3) \quad \max_{r \leq \epsilon_1} |\bar{w}| \geq C_4 \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_5} \left(\oint_{\mathbb{B}(0, (1-\frac{4}{3}a)\epsilon_0)} \bar{w}^2 \right)^{1/2},$$

where \oint denotes the average of the integration.

PROOF. We select a radial cut-off function $\theta \in C_0^\infty(\frac{\epsilon_1}{2} < r < (1 - \frac{11a}{10})\epsilon_0)$ satisfies the properties:

- (i) $\theta(r) = 1$ for $\frac{3\epsilon_1}{4} < r < (1 - \frac{10a}{9})\epsilon_0$.
- (ii) $|\nabla \theta| + |\Delta \theta| \leq C_6$ for $r > (1 - \frac{10a}{9})\epsilon_0$.
- (iii) $|\nabla \theta| \leq C_7 \epsilon_1^{-1}$, $|\Delta \theta| < C_8 \epsilon_1^{-2}$ for $r \leq \frac{3\epsilon_1}{4}$.

From the equation (6.1), we get

$$\Delta(\theta \bar{w}) + b^* \cdot \nabla(\theta \bar{w}) + q^* \theta \bar{w} = \Delta \theta \bar{w} + 2\nabla \theta \cdot \nabla \bar{w} + b^* \cdot \nabla \theta \bar{w}.$$

Assume that $\tau > C$ is large enough. Substituting $u = \theta \bar{w}$ in lemma 1 yields that

$$(6.4) \quad C_2 \tau^3 \int r^\epsilon e^{2\tau\phi(r)} \theta^2 \bar{w}^2 dr d\omega \leq I,$$

where

$$I = \int r^4 e^{2\tau\phi(r)} |\Delta \theta \bar{w} + 2\nabla \theta \cdot \nabla \bar{w} + b^* \cdot \nabla \theta \bar{w}|^2 dr d\omega.$$

Note that $\phi(r)$ is a decreasing function. Furthermore, by the assumptions of $\theta(z)$, we obtain

$$\begin{aligned} I &\leq e^{2\tau\phi(\frac{\epsilon_1}{2})} \int_{\frac{\epsilon_1}{2} < r < \frac{3\epsilon_1}{4}} |\Delta \theta \bar{w} + 2\nabla \theta \cdot \nabla \bar{w} + b^* \cdot \nabla \theta \bar{w}|^2 r dr d\omega \\ &\quad + e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{(1-\frac{10a}{9})\epsilon_0 < r < (1-\frac{11a}{10})\epsilon_0} |\Delta \theta \bar{w} + 2\nabla \theta \cdot \nabla \bar{w} + b^* \cdot \nabla \theta \bar{w}|^2 r dr d\omega. \end{aligned}$$

By standard elliptic estimates, we derive that

$$(6.5) \quad I \leq C_9 e^{2\tau\phi(\frac{\epsilon_1}{2})} \int_{\frac{\epsilon_1}{4} < r < \epsilon_1} \bar{w}^2 r dr d\omega + C_{10} e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{(1-\frac{3a}{2})\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 r dr d\omega.$$

Taking the inequality (6.4) and assumptions of θ into account, we have

$$\begin{aligned}
(6.6) \quad & C_{10} e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{(1-\frac{3a}{2})\epsilon_0 < r < (1-a)\epsilon_0} \bar{w}^2 r \, dr d\omega + C_9 e^{2\tau\phi(\frac{\epsilon_1}{2})} \int_{\frac{\epsilon_1}{4} < r < \epsilon_1} \bar{w}^2 r \, dr d\omega \\
& \geq C_2 \tau^3 \int_{\frac{3\epsilon_1}{4} < r < (1-\frac{10a}{9})\epsilon_0} r^\epsilon e^{2\tau\phi(r)} \bar{w}^2 \, dr d\omega \\
& \geq C_2 \tau^3 \left((1 - \frac{10a}{9})\epsilon_0 \right)^{\epsilon-1} \int_{\frac{3\epsilon_1}{4} < r < (1-\frac{10a}{9})\epsilon_0} e^{2\tau\phi(r)} \bar{w}^2 r \, dr d\omega.
\end{aligned}$$

Since ϵ, ϵ_0 are fixed positive constants, Taking τ large enough, we obtain

$$\frac{1}{2} C_2 \tau^3 \left((1 - \frac{10a}{9})\epsilon_0 \right)^{\epsilon-1} > C_{10}.$$

Taking the hypothesis (6.2) into consideration, we can incorporate the first term in the left hand side of inequality (6.6) into the right hand side. It follows that

$$(6.7) \quad C_9 e^{2\tau\phi(\frac{\epsilon_1}{2})} \int_{\frac{\epsilon_1}{4} < r < \epsilon_1} \bar{w}^2 r \, dr d\omega \geq C_{10} e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{\frac{3\epsilon_1}{4} < r < (1-\frac{10a}{9})\epsilon_0} \bar{w}^2 r \, dr d\omega.$$

Fix such τ , adding the following term

$$e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{r < \frac{3\epsilon_1}{4}} \bar{w}^2 r \, dr d\omega$$

to both sides of last inequality yields that

$$(6.8) \quad e^{2\tau\phi(\frac{\epsilon_1}{2})} \int_{r < \epsilon_1} \bar{w}^2 r \, dr d\omega \geq C_{11} e^{2\tau\phi((1-\frac{10a}{9})\epsilon_0)} \int_{r < (1-\frac{4a}{3})\epsilon_0} \bar{w}^2 r \, dr d\omega,$$

where we have used the fact that ϕ is decreasing. Straightforward calculations show that

$$e^{2\tau(\phi((1-\frac{10a}{9})\epsilon_0) - \phi(\frac{\epsilon_1}{2}))} \geq C_{13} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_{12}}.$$

Thus,

$$(6.9) \quad \int_{r < \epsilon_1} \bar{w}^2 r \, dr d\omega \geq C_{13} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_{12}} \int_{r < (1-\frac{4a}{3})\epsilon_0} \bar{w}^2 r \, dr d\omega.$$

This completes the lemma. \square

Our next goal is to find the relation of lemma 6 and nodal length. We assume that the estimate (6.2) exists. Then the conclusion (6.3) in lemma 6 holds. For $\epsilon_1 \leq \frac{\epsilon}{100}$, if $|z| < \epsilon_1$, using Taylor's expansion gives that

$$|\bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha| \leq \sup_{|z| \leq \epsilon_1} \sup_{|\alpha| = C_5+1} d_1 \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(z) \right|_{\epsilon_1}^{C_5+1},$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial z_1^{\alpha_1}} \frac{\partial}{\partial z_2^{\alpha_2}}$. To control the right hand side of the last inequality, by elliptic estimates and rescaling argument, we have

$$|\bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha| \leq d_2 \left(\int_{\mathbb{B}(0, (1-\frac{4}{3}a)\epsilon_0)} \bar{w}^2 \right)^{1/2} \left(\frac{\epsilon_1}{\epsilon_0} \right)^{C_5+1}.$$

Using the estimate (6.3) in lemma 6, we get

$$|\bar{w}(z) - \sum_{|\alpha| \leq C_5} \frac{1}{\alpha!} \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) z^\alpha| \leq d_3 \left(\frac{\epsilon_1}{\epsilon_0} \right) \max_{|z| \leq \epsilon_1} |\bar{w}|.$$

Choosing ϵ_1/ϵ_0 sufficiently small, by the triangle inequality, we obtain

$$\sup_{|\alpha| \leq C_5} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) \right| \epsilon_1^{|\alpha|} \geq d_4 \max_{|z| \leq \epsilon_1} |\bar{w}|.$$

Applying again the estimate (6.3) to the right hand side of the last inequality yields that

$$(6.10) \quad \sup_{|\alpha| \leq C_5} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(0) \right| \epsilon_0^{|\alpha|} \geq d_5 \left(\int_{\mathbb{B}(0, (1-\frac{4}{3}a)\epsilon_0)} \bar{w}^2 \right)^{1/2}.$$

By standard elliptic estimates, we also have

$$(6.11) \quad \sup_{|z| \leq \frac{\epsilon_0}{2}} \sup_{|\alpha| \leq C_5+1} \left| \frac{\partial^\alpha \bar{w}}{\partial z^\alpha}(z) \right| \epsilon_0^{|\alpha|} \leq d_6 \left(\int_{\mathbb{B}(0, (1-\frac{4}{3}a)\epsilon_0)} \bar{w}^2 \right)^{1/2}.$$

The basic relationship between derivatives and nodal length in two dimensions is shown in [DF2].

LEMMA 7. *Suppose that \bar{w} satisfies (6.10) and (6.11). Then*

$$H^1(z \mid |z| \leq d_7 \bar{\epsilon} \text{ and } \bar{w}(z) = 0) \leq d_8 \bar{\epsilon}.$$

With aid of the last lemma, we can readily obtain an upper nodal length estimate.

PROPOSITION 5. *Let \bar{w} be the solution of (6.1). Suppose that $\bar{\epsilon} \leq \epsilon_0$ and w satisfies the growth condition*

$$(6.12) \quad \int_{(1-\frac{3a}{2})\bar{\epsilon} < r < (1-a)\bar{\epsilon}} \bar{w}^2 \leq C_3 \int_{(1-3a)\bar{\epsilon} < r < (1-\frac{4a}{3})\bar{\epsilon}} \bar{w}^2.$$

Then

$$H^1(z \mid |z| \leq d_9 \bar{\epsilon} \text{ and } \bar{w}(z) = 0) \leq d_{10} \bar{\epsilon}.$$

PROOF. Since the inequalities (6.10) and (6.11) can be derived from (6.12) by lemma 6, the proposition follows from last lemma. \square

7. Total Nodal Length

As the proposition 4 indicates that the eigenfunctions can not grow rapidly on too many small balls. If it grows slowly, we have an upper bound on the local length of nodal sets by proposition 5. In this section, we will link these two arguments together. To achieve it, we will employ a process of repeated subdivision and selection squares. The idea is inspired by [DF2].

Assume that $\mathbb{B}(p, c\lambda^{-\frac{1}{2}})$ is a geodesic ball of double manifold $\overline{\mathcal{M}}$. Choosing c to be small, then it is contained in a conformal chart. Let $w(z) = v(c\lambda^{-\frac{1}{2}}z)$ with c sufficiently small. We know that w satisfies

$$(7.1) \quad \Delta w + \hat{b}(x) \cdot \nabla w + \hat{q}(x)w = 0 \quad \text{in } \mathbb{B}(0, 4).$$

We consider the square $P = \{(x, y) \mid \max(|x|, |y|) \leq \frac{1}{60}\}$ in $\mathbb{B}(0, 4)$ and divide it into a grid of closed square P_l with side $\delta \leq a_1 \lambda^{-1}$. If (5.1) holds for some point $z_l \in P_l$ and for some sufficiently large C_1 . We call P_l as a square of rapid growth. With aid of proposition 4, we are able to obtain the following result.

LEMMA 8. *There are at most $C\lambda^2$ squares with side δ where w is of rapid growth.*

PROOF. Let I_1 be the collection of those indices l for which P_l is a square of rapid growth. For each $l \in I_1$, there exists some point $z_l \in P_l$ such that (5.1) holds. Let $|I_1|$ denote the cardinality of I_1 . Define

$$P_l^* = \{z \mid |z - z_l| < d_1 \delta \lambda^{\frac{1}{2}}\}.$$

The collection of disks P_l^* covers the collection of square P_l for $l \in I_1$. We choose a maximal collection of disjoint disks of P_l^* and denote it as I_2 . If $l \in I_2$, we define

$$P_l^{**} = \{z \mid |z - z_l| < 4d_1 \delta \lambda^{\frac{1}{2}}\}.$$

Since the collection of disks in I_2 are disjoint and maximal, we obtain that

$$\bigcup_{l \in I_2} P_l^{**} \supseteq \bigcup_{l \in I_1} P_l^* \supseteq \bigcup_{l \in I_1} P_l.$$

Thus,

$$|I_2| \times 16d_1^2 \delta^2 \lambda \geq |I_1| \delta^2,$$

which implies

$$|I_2| \lambda \geq d_2 |I_1|.$$

Recall from proposition 4 that $|I_2| \leq d_3 \lambda$. Therefore, we obtain the desirable estimate $|I_1| \leq d_4 \lambda^2$. □

Now we introduce a iterative process of bisecting squares. We begin by dividing the square into a grid of square $P_l(1)$ with side $\delta(1) = a_1 \lambda^{-1}$, then separate them into two categories $R_l(1)$ and $S_l(1)$. $R_l(1)$ are those where w is of rapid growth and $S_l(1)$ are those where (5.1) fails for w . We continue to bisect each square $R_l(1)$ to obtain square $P_l(2)$ with side $\delta(2) = \frac{\delta(1)}{2}$. Again, we split $P_l(2)$ into the subcollection $R_l(2)$ with rapid growth and $S_l(2)$ with slow growth. We repeat the process at each step k . Then there are squares $R_l(k)$ and $S_l(k)$ with $\delta(k) = \frac{\delta(1)}{2^k}$. We count the number of $R_l(k)$ and $S_l(k)$ at step k .

LEMMA 9. (i) *The number of squares $R_l(k)$ is at most $C_2 \lambda^2$.*
(ii) *The number of squares $S_l(k)$ is at most $C_3 \lambda^2$.*

PROOF. The conclusion (i) follows directly from the lemma 8. We only need to show (ii). If $k = 1$, the conclusion (ii) follows because the total number of squares is at most the order of λ^2 . If $k \geq 2$, by construction of those squares,

$$|S_l(k)| \leq 4|R_l(k-1)| \leq C_4 \lambda^2,$$

where we have used (i) in the last inequality. The lemma is done. □

Next lemma tells that almost every point lies in some $R_l(k)$ with slow growth. It is the lemma 6.3 in [DF2].

LEMMA 10. $\bigcup_{k,l} S_l(k)$ covers the square P except for singular points $\mathcal{S} = \{z \in P \mid w(x) = 0, \nabla w = 0\}$.

We are ready to give the proof of Theorem 3.

PROOF OF THEOREM 3. Consider $\bar{w}(z) = w(z_l + \epsilon_0^{-1}\delta(k)z)$. Then $\bar{w}(z)$ satisfies the equation (6.1). Choosing a finite collection of $z_l \in S_l(k)$ and applying proposition 5, we have

$$(7.2) \quad H^1(z | w(z) = 0 \text{ and } z \in S_l(k)) \leq C_5 2^{-k} \lambda^{-1}.$$

Furthermore, thanks to lemma 10,

$$(7.3) \quad \begin{aligned} H^1(z | w(z) = 0 \text{ and } \max(|x|, |y|) \leq \frac{1}{60}) &\leq \sum_{l,k} H^1(z | w(z) = 0 \text{ and } z \in S_l(k)) \\ &\leq \lambda^2 \sum_k C_5 2^{-k} \lambda^{-1} \\ &\leq C_6 \lambda, \end{aligned}$$

where we have used (ii) in lemma 9 and (7.2). Since $w(z) = v(c\lambda^{-\frac{1}{2}}z)$, by the rescaling argument, we obtain

$$H^1(\{v(z) = 0\} \cap \mathbb{B}(p, c\lambda^{-\frac{1}{2}})) \leq C_6 \lambda^{\frac{1}{2}}.$$

Finally, covering $\overline{\mathcal{M}}$ with order λ of geodesic balls with radius $c\lambda^{-\frac{1}{2}}$, we readily deduce that

$$H^1(z \in \overline{\mathcal{M}} | v(z) = 0) \leq C_7 \lambda^{\frac{3}{2}}.$$

So is $H^1(\mathcal{N}_\lambda)$. □

References

- [BC] L. Bakri and J.B. Casteras, Quantitative uniqueness for Schrödinger operator with regular potentials, *Math. Methods Appl. Sci.* 37(2014), 1992-2008.
- [Br] J. Brüning, Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators, *Math. Z.* 158(1978), 15-21.
- [BL] K. Bellova and F.H. Lin, Nodal sets of Steklov eigenfunctions, arXiv:1402.4323.
- [C] A.P. Calderón, On a inverse boundary value problem, in “Seminar in Numerical Analysis and Its Applications to Continuum Physics”, 1980, 65-73, Soc. Brasileira de Matemática, Rio de Janeiro.
- [CM] T.H. Colding and W. P. Minicozzi II, Lower bounds for nodal sets of eigenfunctions, *Comm. Math. Phys.* 306(2011), 777-784.
- [D] R-T Dong, Nodal sets of eigenfunctions on Riemann surfaces, *J. Differential Geom.* 36(1992), 493-506.
- [DF] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, *Invent. Math.* 93(1988), 161-183.
- [DF1] H. Donnelly and C. Fefferman, Nodal sets of eigenfunctions: Riemannian manifolds with boundary, in: *Analysis, Et Cetera*, Academic Press, Boston, MA, 1990, 251-262.
- [DF2] H. Donnelly and C. Fefferman, Nodal sets for eigenfunctions of the Laplacian on surfaces, *J. Amer. Math. Soc.* 3(1990), no. 2, 333-353.
- [GP] A. Girouard and I. Polterovich, Spectral geometry of the Steklov problem, arXiv:1411.6567.
- [HL] Q. Han and F.-H. Lin, Nodal sets of solutions of Elliptic Differential Equations, book in preparation (online at <http://www.nd.edu/qhan/nodal.pdf>).
- [HS] R.S. Hardt and L. Simon, Nodal sets of solutions of elliptic equations, *J. Differential Geom.* 30(1989), 505-522.
- [HSo] H. Hezari and C.D. Sogge, A natural lower bound for the size of nodal sets, *Anal. PDE.* 5(2012), no. 5, 1133-1137.
- [Lin] F.-H. Lin, Nodal sets of solutions of elliptic equations of elliptic and parabolic equations, *Comm. Pure Appl Math.* 44(1991), 287-308.

- [PST] I. Polterovich, D. Sher and J. Toth, Nodal length of Steklov eigenfunctions on real-analytic Riemannian surfaces, arXiv:1506.07600.
- [SWZ] C.D. Sogge, X. Wang and J. Zhu, Lower bounds for interior nodal sets of Steklov eigenfunctions, arXiv:1503.01091.
- [SZ] C.D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets, Math. Res. Lett. 18(2011), 25-37.
- [SZ1] C.D. Sogge and S. Zelditch, Lower bounds on the Hausdorff measure of nodal sets II, Math. Res. Lett. 19(2012), no.6, 1361-1364.
- [WZ] X. Wang and J. Zhu, A lower bound for the nodal sets of Steklov eigenfunctions, to appear in Mathematical Research Letters.
- [Z] S. Zelditch, Local and global analysis of eigenfunctions on Riemannian manifolds, in: Handbook of Geometric Analysis, in: Adv. Lect. Math. (ALM), vol. 7(1), Int. Press, Somerville, MA, 2008, 545-658.
- [Z1] S. Zelditch, Measure of nodal sets of analytic steklov eigenfunctions, arXiv:1403.0647.
- [Zh] J. Zhu, Doubling property and vanishing order of Steklov eigenfunctions, Comm. Partial Differential Equations 40(2015), no. 8, 1498-1520.

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA,
EMAILS: JZHU43@MATH.JHU.EDU